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Original Research Article

EFFICIENCY BALANCED DESIGNS

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Abstract: This paper provides the methods of construction of binary and non-binary efficiency balanced (EB) designs. The methods are based on the incidence matrices of balanced incomplete block (BIB) designs and group divisible (GD) designs.

Key Words: BIB design, GD design, EB design, binary and non-binary designs, augmented design, kronecker product of matrices.

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1. Introduction: The concept of an EB design has been introduced by Jones (1959). However, he has called such design as a total balanced design. Later Puri and Nigam (1975a) have renamed this total balanced design of Jones (1959) as efficiency balanced (EB) design. Calinski (1971) has discussed the concept of EB design in detail and has given some examples of EB designs by trial and error. Puri and Nigam (1975a) have proved that a design is an EB design if all the off-diagonal elements of the matrix $P = NK^{-1}N'$ are all proportional to the product of the two relevant replications. Puri and Nigam (1975b) have given a systematic procedure of construction of EB designs by merging of treatments in BIB and EB design. Several authors have studied the concept of EB design and have given many interesting properties and methods of construction of EB designs. To quote we mention few Williams (1975), Dey and Singh (1980), Kageyama (1980), Dey et al. (1981), Ghosh and Karmarkar (1988).

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In what follows, we denote by $\otimes$ the kronecker product of matrices, $\mathbf{1}'_x$ the $1 \times x$ row vector of ones, $\mathbf{1}_x$ the column $x \times 1$ column vector of ones, $\mathbf{1}'_x \otimes N$ the x replications of N, I_x the identity matrix of order x, $O_{x\times y}$ the null matrix of order $x \times y$, $J_{x\times y}$ the matrix of ones of order $x \times y$, and by x_l (l = 1, 2), p, q, s, t the positive integers.

2. Methods of Construction: In this section, we describe methods of construction of binary and non-binary EB designs making use of the incidence matrices of some known BIB designs and GD designs.

Theorem 2.1: Let N_1 be the $v_1 \times b_1$ incidence matrix of a BIB design D_1 with parameters $v_1, b_1, r_1, k_1, \lambda_1$. Then

$$N = \begin{bmatrix} N_1 & J_{v_1 \times q} & I_{v_1} \\ \mathbf{1}'_{b_1} & \boldsymbol{O}'_q & \mathbf{1}'_{v_1} \end{bmatrix}$$

is the incidence matrix of a binary EB design D with parameters $v = v_1 + 1$, $b = b_1 + q + v_1$, $r' = \{(r_1 + q + 1)\mathbf{1}'_{v_1}, (b_1 + v_1)\}$, $k' = \{(k_1 + 1)\mathbf{1}'_{b_1}, v_1\mathbf{1}'_q, 2\mathbf{1}'_{v_1}\}$ and
$E = \{\lambda_1 v_1 + q(k_1 + 1)\}\{v_1(r_1 + q + 1) + (b_1 + v_1)\}/v_1(k_1 + 1)(r_1 + q + 1)^2$ if and only if constant q satisfy the equality
$q\{2b_1 + v_1(k_1 + 1)\} = v_1\{r_1(2r_1 + k_1 + 3) + (k_1 + 1) - 2\lambda_1(b_1 + v_1)\}$.

Proof: Evidently the off-diagonal elements of the $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{\lambda_1}{(k_1 + 1)} + \frac{q}{v_1} \qquad ; i, j \leq v_1 \;\&\; i \neq j \tag{2.1}$$

$$c_{ij} = \frac{r_1}{(k_1 + 1)} + \frac{1}{2} \qquad ; i \leq v_1 \;\&\; j = v_1 + 1 \tag{2.2}$$

Now suppose D is an EB design. Then one has $c_{ij} = c r_i r_j$. Hence, considering the c_{ij}'s as given above we have

$$\frac{\lambda_1}{(k_1 + 1)} + \frac{q}{v_1} = c(r_1 + q + 1)^2 \tag{2.3}$$

$$\frac{r_1}{(k_1 + 1)} + \frac{1}{2} = c(r_1 + q + 1)(b_1 + v_1) \tag{2.4}$$

Now from (2.3) and (2.4) eliminating c we get the required result. Using (2.3) we get efficiency E. Hence the proof.

Table 2.1: EB designs using Theorem 2.1

Sr. No.	Series No.	Parameters $v_1, b_1, r_1, k_1, \lambda_1$	q	Parameters of EB design v, b, r', k'	E
1	8	6,6,5,5,4	0	7,12, $\{6\mathbf{1}'_6, 12\}$, $\{6\mathbf{1}'_6, 6\mathbf{1}'_0, 2\mathbf{1}'_6\}$	0.89
2	10	7,7,3,3,1	2	8,16, $\{6\mathbf{1}'_7, 14\}$, $\{4\mathbf{1}'_7, 7\mathbf{1}'_2, 2\mathbf{1}'_7\}$	0.83
3	19	9,18,8,4,3	3	10,30, $\{12\mathbf{1}'_9, 27\}$, $\{5\mathbf{1}'_{18}, 9\mathbf{1}'_3, 2\mathbf{1}'_9\}$	0.88
4	25	10,30,9,3,2	6	11,46, $\{16\mathbf{1}'_{10}, 40\}$, $\{4\mathbf{1}'_{30}, 10\mathbf{1}'_6, 2\mathbf{1}'_{10}\}$	0.86
5	27	10,15,9,6,5	0	11,25, $\{10\mathbf{1}'_{10}, 25\}$, $\{7\mathbf{1}'_{15}, 10\mathbf{1}'_0, 2\mathbf{1}'_{10}\}$	0.89
6	29	11,11,5,5,2	1	12,23, $\{7\mathbf{1}'_{10}, 22\}$, $\{6\mathbf{1}'_{11}, 11\mathbf{1}'_1, 2\mathbf{1}'_{11}\}$	0.86

Theorem 2.2: Let N_1 be the $v_1 \times b_1$ incidence matrix of a BIB design D_1 with parameters v_1, b_1, $r_1 = v_1 - 1$, $k_1 = 2\lambda_1$, λ_1. Then

$$N = \begin{bmatrix} \mathbf{1}'_p \otimes N_1 & & \mathbf{1}'_{tq} \otimes I_{v_1} & & & O_{v_1 \times q} \\ \boldsymbol{O}'_{pb_1} & x_1\mathbf{1}'_{qv_1} & \boldsymbol{O}'_{qv_1} & \cdots & \boldsymbol{O}'_{qv_1} & x_2\mathbf{1}'_q \\ \boldsymbol{O}'_{pb_1} & \boldsymbol{O}'_{qv_1} & x_1\mathbf{1}'_{qv_1} & \cdots & \boldsymbol{O}'_{qv_1} & x_2\mathbf{1}'_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O}'_{pb_1} & \boldsymbol{O}'_{qv_1} & \boldsymbol{O}'_{qv_1} & \cdots & x_1\mathbf{1}'_{qv_1} & x_2\mathbf{1}'_q \end{bmatrix}$$

is the incidence matrix of an EB design D with parameters $v = v_1 + t$, $b = pb_1 + tqv_1 + q$, $r' = \{(pr_1 + tq)\mathbf{1}'_{v_1}, q(x_1v_1 + x_2)\mathbf{1}'_t\}$, $k' = \{k_1\mathbf{1}'_{pb_1}, (x_1 + 1)\mathbf{1}'_{tqv_1}, x_2t\mathbf{1}'_q\}$ and $E = p\lambda_1\{v_1(pr_1 + tq) + tq(x_1v_1 + x_2)\}/k_1(pr_1 + tq)^2$ if and only if t satisfy the equality $tqx_1{}^2k_1 = p\lambda_1x_2(x_1 + 1)^2$.

Proof: Evidently the off-diagonal elements of the $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{p\lambda_1}{k_1} \quad ; i,j \le v_1 \ \& \ i \ne j \tag{2.5}$$

$$c_{ij} = \frac{qx_1}{(x_1 + 1)} \quad ; i \le v_1 \ \& \ j \ge v_1 + 1 \tag{2.6}$$

$$c_{ij} = \frac{qx_2}{t} \quad ; j \ge v_1 + 1 \ \& \ i \ne j \tag{2.7}$$

Now suppose D is an EB design. Then one has $c_{ij} = cr_ir_j$. Hence, considering the $c_{ij}'s$ as given above we have

$$\frac{p\lambda_1}{k_1} = c(pr_1 + tq)^2 \tag{2.8}$$

$$\frac{qx_1}{(x_1 + 1)} = c(pr_1 + tq)q(x_1v_1 + x_2) \tag{2.9}$$

$$\frac{qx_2}{t} = c\{q(x_1v_1 + x_2)\}^2 \tag{2.10}$$

Now from (2.8), (2.9) and (2.10) eliminating c we get the required result. Using (2.8) we get efficiency E. Hence the proof.

Corollary 2.1: If we take $x_1 = x_2 = q = \lambda_1$ and $p = \frac{t}{2}$, then the design given in Theorem 2.2 is an EB design D with parameters $v = v_1 + t$, $b = \frac{tb_1}{2} + tv_1\lambda_1 + \lambda_1$, $r' = \left\{t(\frac{r_1}{2} + \lambda_1)\mathbf{1}'_{v_1}, \lambda_1{}^2(v_1 + 1)\mathbf{1}'_t\right\}$, $k' = \left\{k_1\mathbf{1}'_{\frac{tb_1}{2}}, (\lambda_1 + 1)\mathbf{1}'_{tv_1\lambda_1}, t\lambda_1\mathbf{1}'_{\lambda_1}\right\}$ and $E = \lambda_1\{v_1(r_1 + 2\lambda_1) + 2\lambda_1{}^2(v_1 + 1)\}/k_1(r_1 + 2\lambda_1)^2$.

Table 2.2: EB designs using Corollary 2.1

Sr. No.	Series No.	Parameters $v_1,b_1,r_1,k_1,\lambda_1$	x_1	x_2	p	q	s	Parameters of EB design v, b, r', k'	E
1		3,3,2,2,1	1	1	2	1	4	7,19, $\{8\mathbf{1}'_3, 4\mathbf{1}'_4\}$, $\{2\mathbf{1}'_6, 2\mathbf{1}'_{12}, 4\mathbf{1}'_1\}$	0.63
2	1	4,6,3,2,1	1	1	8	1	16	20,113, $\{40\mathbf{1}'_4, 5\mathbf{1}'_{16}\}$, $\{2\mathbf{1}'_{48}, 2\mathbf{1}'_{64}, 16\mathbf{1}'_1\}$	0.6
3	3	5,10,4,2,1	1	1	8	1	16	21,161, $\{48\mathbf{1}'_5, 6\mathbf{1}'_{16}\}$, $\{2\mathbf{1}'_{80}, 2\mathbf{1}'_{80}, 16\mathbf{1}'_1\}$	0.58
4	18	9,36,8,2,1	1	1	4	1	8	17,217, $\{40\mathbf{1}'_9, 10\mathbf{1}'_8\}$, $\{2\mathbf{1}'_{144}, 2\mathbf{1}'_{72}, 8\mathbf{1}'_1\}$	0.55

Corollary 2.2: If we take $x_1 = x_2 = \lambda_1$, $q = k_1$ and $p = t$, then the design given in Theorem 2.2 is an EB design D with parameters $v = v_1 + t$, $b = tb_1 + tv_1k_1 + k_1$, $r' = \{t(r_1 + k_1)\mathbf{1}'_{v_1}, k_1\lambda_1(v_1 + 1)\mathbf{1}'_t\}$, $k' = \{k_1\mathbf{1}'_{tb_1}, (\lambda_1 + 1)\mathbf{1}'_{tv_1k_1}, t\lambda_1\mathbf{1}'_{k_1}\}$ and. $E = \lambda_1\{v_1(r_1 + k_1) + k_1\lambda_1(v_1 + 1)\}/k_1(r_1 + k_1)^2$.

Table 2.3: EB designs using Corollary 2.2

Sr. No.	Series No.	Parameters $v_1, b_1, r_1, k_1, \lambda_1$	x_1	x_2	p	q	s	Parameters of EB design v, b, $\boldsymbol{r}'$, $\boldsymbol{k}'$	E
1	6	6,15,5,2,1	1	1	16	2	16	22,434, $\{112\mathbf{1}'_6, 14\mathbf{1}'_{16}\}$, $\{2\mathbf{1}'_{240}, 2\mathbf{1}'_{192}, 16\mathbf{1}'_2\}$	0.57
2	12	7,21,6,2,1	1	1	4	2	4	11,142, $\{32\mathbf{1}'_7, 16\mathbf{1}'_4\}$, $\{2\mathbf{1}'_{84}, 2\mathbf{1}'_{56}, 4\mathbf{1}'_2\}$	0.56
3	14	8,28,7,2,1	1	1	3	2	3	11,134, $\{27\mathbf{1}'_8, 18\mathbf{1}'_3\}$, $\{2\mathbf{1}'_{84}, 2\mathbf{1}'_{48}, 3\mathbf{1}'_2\}$	0.56
4	24	10,45,9,2,1	1	1	4	2	4	14,262, $\{44\mathbf{1}'_{10}, 22\mathbf{1}'_4\}$, $\{2\mathbf{1}'_{180}, 2\mathbf{1}'_{80}, 4\mathbf{1}'_2\}$	0.55
5	31	11,55,10,2,1	1	1	9	2	9	20,695, $\{108\mathbf{1}'_{11}, 24\mathbf{1}'_9\}$, $\{2\mathbf{1}'_{495}, 2\mathbf{1}'_{198}, 9\mathbf{1}'_2\}$	0.54

Theorem 2.3: Let N_1 be the incidence matrix of a GD design D_1 with parameters v_1, b_1, r_1, k_1, m, n, $\lambda_1 = 0$, λ_2. Then

$$N = \begin{bmatrix} \mathbf{1}'_p \otimes N_1 & J_{n\times q} \otimes I_m & J_{n\times q} \otimes I_m & \cdots & J_{n\times q} \otimes I_m & O_{v_1 \times q} \\ \mathbf{O}'_{pb_1} & x_1\mathbf{1}'_{qm} & \mathbf{O}'_{qm} & \cdots & \mathbf{O}'_{qm} & x_2\mathbf{1}'_q \\ \mathbf{O}'_{pb_1} & \mathbf{O}'_{qm} & x_1\mathbf{1}'_{qm} & \cdots & \mathbf{O}'_{qm} & x_2\mathbf{1}'_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O}'_{pb_1} & \mathbf{O}'_{qm} & \mathbf{O}'_{qm} & \cdots & x_1\mathbf{1}'_{qm} & x_2\mathbf{1}'_q \end{bmatrix}$$

is the incidence matrix of an EB design D with parameters $v = v_1 + t$, $b = pb_1 + tqm + q$, $\boldsymbol{r}' = \{(pr_1 + tq)\mathbf{1}'_{v_1}, q(x_1 m + x_2)\mathbf{1}'_t\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}'_{pb_1}, (x_1 + n)\mathbf{1}'_{tqm}, x_2 t\mathbf{1}'_q\}$ and $E = p\lambda_2\{v_1(pr_1 + tq) + tq(x_1 m + x_2)\}/k_1(pr_1 + tq)^2$ if and only if t, p, q satisfy the equalities $tx_1{}^2 = x_2(x_1 + n)^2$ and $p\lambda_2(x_1 + n) = qk_1$.

Proof: Evidently the off-diagonal elements of the $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{q}{(x_1 + n)} \qquad ; i, j \le v_1 \;\&\; i \ne j \qquad (2.11)$$

$$c_{ij} = \frac{p\lambda_2}{k_1} \qquad ; i, j \le v_1 \;\&\; i \ne j \qquad (2.12)$$

$$c_{ij} = \frac{qx_1}{(x_1 + n)} \qquad ; i \le v_1 \;\&\; j \ge v_1 + 1 \qquad (2.13)$$

$$c_{ij} = \frac{qx_2}{t} \qquad ; j \ge v_1 + 1 \;\&\; i \ne j \qquad (2.14)$$

Now suppose D is an EB design. Then one has $c_{ij} = cr_ir_j$. Hence, considering the c_{ij}'s as given above we have

$$\frac{q}{(x_1 + n)} = c(pr_1 + tq)^2 \qquad (2.15)$$

$$\frac{p\lambda_2}{k_1} = c(pr_1 + tq)^2 \qquad (2.16)$$

$$\frac{qx_1}{(x_1 + n)} = c(pr_1 + tq)q(x_1 m + x_2) \qquad (2.17)$$

$$\frac{qx_2}{t} = c\{q(x_1 m + x_2)\}^2 \qquad (2.18)$$

Now from (2.15), (2.16), (2.17) and (2.18) eliminating c we get the required result. Using (2.16) we get efficiency *E*. Hence the proof.

Table 2.4: EB designs using Theorem 2.3

Sr. No.	Series No.	Parameters $v_1, b_1, r_1, k_1, m, n, \lambda_1, \lambda_2$	x_1	x_2	p	q	t	Parameters of EB design v, b, r', k'	E
1	SR1	4,4,2,2,2,2,0,1	2	2	1	2	2	6,14, $\{6\mathbf{1}'_4, 12\mathbf{1}'_2\}$, $\{2\mathbf{1}'_4, 4\mathbf{1}'_8, 4\mathbf{1}'_2\}$	0.67
2	SR3	4,12,6,2,2,2,0,3	2	2	1	6	2	6,42, $\{18\mathbf{1}'_4, 36\mathbf{1}'_2\}$, $\{2\mathbf{1}'_{12}, 4\mathbf{1}'_{24}, 4\mathbf{1}'_6\}$	0.67
3	SR4	4,16,8,2,2,2,0,4	2	2	2	16	2	6,112, $\{48\mathbf{1}'_4, 96\mathbf{1}'_2\}$, $\{2\mathbf{1}'_{32}, 4\mathbf{1}'_{64}, 4\mathbf{1}'_{16}\}$	0.67
4	SR22	6,20,10,3,3,2,0,5	2	3	3	20	3	9,260, $\{90\mathbf{1}'_6, 180\mathbf{1}'_3\}$, $\{3\mathbf{1}'_{60}, 4\mathbf{1}'_{180}, 9\mathbf{1}'_{20}\}$	0.67
5	R29	8,24,6,2,4,2,0,1	2	2	1	2	2	10,42, $\{10\mathbf{1}'_8, 20\mathbf{1}'_2\}$, $\{2\mathbf{1}'_{24}, 4\mathbf{1}'_{16}, 4\mathbf{1}'_2\}$	0.6
6	R199	26,26,9,9,13,2,0,3	1	2	2	2	6	32,210, $\{30\mathbf{1}'_{26}, 30\mathbf{1}'_6\}$, $\{9\mathbf{1}'_{52}, 3\mathbf{1}'_{156}, 12\mathbf{1}'_2\}$	0.71

Remark 2.2: In Table 2.1 to Table 2.3 series numbers are according to Raghavrao (1971), page 91 and in Table 2.4 series numbers of GD designs are according to Clatworthy (1973).

3. Conclusion: Here the new construction methods of efficiency balanced designs are given. The constructed methods are flexible enough to incorporate number of incidence matrices of GD and BIB designs. The designs so constructed are found to have applications in industrial, agricultural and pharmaceutical experiments.

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