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Original Research Article

FIXED POINT THEOREMS FOR *T*-ZAMFIRESCU MAPPING ON CONE METRIC SPACES

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Abstract*:* In this paper, we obtain sufficient conditions for the existence of a unique fixed point of T-Zamfirescuin complete cone metric spaces and we introduce T-Manniteration and study the convergence of these iterations for the class of T-Zamficescu operators in real Banach spaces.

Keywords:-Cone Metric Space, T-Zamfirescu mapping , cone normed space .

Introduction*:* In [8], Huang and Chang gave the notion of cone metric space, replacing the set of real numbers by ordered Banach Space and introduced some fixed point theorems for function satisfying contractive conditions in Banach Spaces. Sh. Rezapour and R. Hamalbarani [12] were generalized result of [8] by omitting the normality condition, which is milestone in developing fixed point theory in cone metric space. After that several articles on fixed point theorems in cone metric space were obtained by different mathematicians such as M. Abbas , G. Junck[9] , D. Ilic [2] etc

A. Beiranvand etc [1] introduced the T−contraction and T−contractive mappings and then they extended the Banach contraction principle and the Edelstein's fixed point Theorem.

The T−Kannan contractive mappings introduced by S. Moradi [13], and extend in this way the Kannan's fixed point theorem [10]. The corresponding version of T-contractive, T-Kannan mappings and T−Chalterjea contractions on cone metric spaces was studied in [4] and [5] respectively, obtained sufficient conditions for the existence of a unique fixed point of these mappings in complete cone metric spaces. In [6] they studied the existence of fixed points for T-Zamficescu operators in complete metric spaces and proved aconvergence theorem of T-Picard iteration for the class of T-Zamficescu operators.

In analysis of these facts, thus the purpose of this paper is to study the existence of fixed points of T−Zamficescu defined on a complete cone metric space(X, d), generalizing consequently the results given in [3] and [14],and we introduce

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T-Manniteration and establish strong convergence theorems of these iteration schemes to the fixed point of T-Zamficescu operators inreal Banach spaces.

Preliminaries & Definition

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A subset $P \subseteq E$is said to be a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$

(ii) $a, b \in R,\ \ a, b \geq 0,\ x, y \in P$ implies $ax + by \in P$

(iii) $P \cap (-P) = \{0\}$

For a given cone P subset of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if$y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in intP$ where $intP$ denotes interior of **P** and is assumed to be nonempty.

Definition 2.2. [7] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x,y)$ for every $x, y \in X,\ d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$ for every $x, y \in X$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for every $x, y, z \in X$.

Then d is a cone metric on X and *(X, d)* is a cone metric space.

Example **2.3**[3] Let $E = R^n$,$P = \{(x,y) \in E : x,y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.3. Let E be a Banach space and P ⊂ E a cone. The cone Pis called normal if there is a number K > 0 such that for all $x, y \in E$,

$$0 \leq x \leq y \text{Implies} \|x\| \leq K\|y\|$$

The least positive number satisfying the aboveis called the normal constant of P.

Definition 2.4. [11] Let X be a vector space over R. Suppose the mapping$\|\cdot\| :\ X \to E$ satisfies

(i) $\|x\| \geq 0$ for all $x \in X$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii)$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

(iv)$\|kx\| = |\,k\,|\,\|x\|$ for all $k \in R$.

Then $\|\cdot\|$ is called a norm on X, and $(X, \|\cdot\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with metric defined by $d(x,y) = \|x - y\|$

Definition 2.5. [3]Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$converges to x if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n,x) \leq c$ for all $n \geq N$

We shall denote it by$\lim_{n\to\infty} x_n = x$ or$x_n \to x$.

(ii) $\{x_n\}$ is a Cauchy sequence, if for every $c \in E$ with $0 \ll c$ there is a natural number N such that

$d(x_n,x) \leq c$ for all $n,m \geq N$

(iii)$(X\,,d)$ is a complete cone metric space if every Cauchy sequence is convergent in X.

Definition 2.6. [11] Let $(X, \|\cdot\|)$ be a cone normed space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$converges to x if for every $c \in E$ with $0 \ll c$there is a natural number N such that $\|x_n - x\| \leq c$ for all $n \geq$

We shall denote it by$\lim_{n\to\infty} x_n = x$ or$x_n \to x$.

(ii) $\{x_n\}$ is a Cauchy sequence, if for every $c \in E$ with $0 \ll c$ there is a natural number N such that

$\|x_n - x_m\| \leq c$ for all $n,m \geq N$

.

(iii)$(X\,, \|\cdot\|)$ is a complete cone normed space if every Cauchy sequence is convergent. A complete cone normed space is called a Cone Banach space.

Lemma 2.7. [3] Let (X, d) be a cone normed space. P be a normal cone with constant K. Let $\{x_n\}$, $\{y_n\}$ be a sequence in X and $x, y \in X$ Then

(i) $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$

(iii) If $\{x_n\}$ is a Cauchy sequence if and only $\lim_{n,m\to\infty} d(x_n, x_m) = 0$

(iv) If the $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then $d(x_n, y_n) \to d(x, y)$

Definition 2.8. Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T : X \to X$. Then

(i) T is said to be continuous, if $\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} T(x_n) = T(x)$ for all $\{x_n\}$ and x in X.

(ii) T is said to be sub-sequentially convergent if we have, for every sequence $\{y_n\}$, if $T(y_n)$ is convergent, then $\{y_n\}$ has a convergent sub-sequence.

(iii) T is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $T(y_n)$ is convergent then $\{y_n\}$ also is convergent.

Now, following the ideas of T. Zamfirescu [D] we introduce the notion of T−Zamfirescu mappings.

Definition 2.9. [14] Let (X, d) be a cone metric space and $T, S : X \to X$ two mappings. S is called a T−Zamfirescu mapping, (TZ-mapping), if and only if, there are real numbers, $0 \le a < 1$, $0 \le b, c < 1/2$ such that for all $x, y \in X$, at least one of the next conditions are true:

(TZ_1): $d(TSx, TSy) \le a d(Tx, Ty)$.

(TZ_2): $d(TSx, TSy) \le b[d(Tx, TSx) + d(Ty, TSy)]$.

(TZ_3): $d(TSx, TSy) \le c[d(Tx, TSy) + d(Ty, TSx)]$.

Definition 2.10. Let E be a Banach space, $x_0 \in E$ and $T, S: E \to E$ be two mappings. The sequence $\{Tx_n\} \in E$ defined by

$$Tx_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n TSx_n \qquad n = 1,2,3 \ldots, \tag{2.1}$$

where $\{\alpha_n\} \in [0,1]$ is called the T-Mann iteration associated to S.

Lemma 2.11. [15] Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be sequences of non negative numbers satisfying the inequality

$r_{n+1} \le (1 - s_n)r_n + s_n t_n$ for all $n \ge 1$

If $\sum_{n=1}^{\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = 0$ then $\lim_{n\to\infty} r_n = 0$

1. Main Result:-

Lemma 3.1: Let (X, d) be a cone metric space and $T, S : X \to X$ two mappings with

$$d(TSx, TSy) \le b[d(Tx, TSy) + d(Ty, TSx) - d(TSy, TSx)] \qquad \ldots(3.1)$$

for all $x, y \in X$. where $0 \le a \le 1$ and $0 \le b \le 1$. Then S is a T−Zamfirescu mapping.

Proof:- For all $x, y \in X$,

$$d(TSx, TSy) \le b[d(Tx, TSy) + d(Ty, TSx) - d(TSy, TSx)]$$
$$\le b[d(Tx, TSx) + d(TSx, TSy) + d(Ty, TSy) + d(TSy, TSx) - d(TSy, TSx)]$$
$$\le b[d(Tx, TSx) + d(TSx, TSy) + d(Ty, TSy)]$$
$$d(TSx, TSy) \le b[d(Tx, TSx) + d(TSx, TSy) + d(Ty, TSy)]$$
$$d(TSx, TSy) - b\, d(TSx, TSy) \le b[d(Tx, TSx) + d(Ty, TSy)]$$
$$(1 - b)d(TSx, TSy) \le b[d(Tx, TSx) + d(Ty, TSy)]$$
$$d(TSx, TSy) \le \frac{b}{(1 - b)}[d(Tx, TSx) + d(Ty, TSy)] \qquad \ldots(3.2)$$

Therefore by denoting $\lambda = \frac{b}{1-b}$

We have $0 < \lambda < 1$. Hence for all $x, y \in X$ the following inequality holds

$d(TSx, TSy) \leq \lambda[d(Tx, TSx) + d(Ty, TSy)]$

Thus, S is a T–Zamfirescu mapping.

Theorem 3.2:-Let (X, d) be a complete cone metric space, P is normal cone with normal cone with normal constant K. Moreover, let $T: X \to X$ be a continuous and one to one mapping and $S: X \to X$ a continuous mapping with

$d(TSx, TSy) \leq b[d(Tx, TSy) + d(Ty, TSx) - d(TSy, TSx)]$

Then

(i) For every $x_0 \in X$

$\lim_{n \to \infty} d(TS^{n+1}x_0, TS^n x_0) = 0$

(ii) There is $y_0 \in X$ such that

$\lim_{n \to \infty} TS^n x_0 = y_0$

(iii)If T is sub-sequentially convergent, then $\{S^n x_0\}$ has a convergent sub sequence.

(iv) There is a unique $z_0 \in X$ such that $Sz_0 = z_0$.

(v) If T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ convergent to z_0

Proof:-

(i)By lemma (3.1) S is a T–Zamfirescu mapping. Therefore, ∃ a real number $0 \leq h < 1$ such that

$d(TSx, TSy) \leq h\, d(Tx, Ty)$ for all $x, y \in X$

Suppose $x_0 \in X$ is an arbitrary point and the Picard iteration associated to S, $\{x_n\}$ is defined by

$x_{n+1} = Sx_n = S^n x_0 \qquad n = 0,1,2 \ldots$

Thus,

$$d(TS^{n+1}x_0, TS^n x_0) \leq h\, d(TS^n x_0, TS^{n-1}x_0)$$
$$\leq h[h\, d(TS^{n-1}x_0, TS^{n-2}x_0)]$$
$$\leq h^2 d(TS^{n-1}x_0, TS^{n-2}x_0)$$
$$\leq h^3[d(TS^{n-2}x_0, TS^{n-3}x_0)]$$

Continue to n times, for all n we have

$$d(TS^{n+1}x_0, TS^n x_0) \leq h^n[d(TSx_0, Tx_0)] \quad \ldots(3.3)$$

Form the above, and fact the cone P is normal cone we obtain that

$\|d(TS^{n+1}x_0, TS^n x_0)\| \leq Kh^n \|d(TSx_0, Tx_0)\|$

Taking limit $n \longrightarrow \infty$ in the above inequality we can conclude that

$\lim_{n \to \infty} d(TS^{n+1}x_0, TS^n x_0) = 0$

(ii) Now, for $m, n \in N$ with $m > n$ from (3.3) we get

$$d(TS^m x_0, TS^n x_0) \leq (h^n + \ldots\ldots + h^{m-1}) d(TSx_0, Tx_0)$$
$$\leq \frac{h^n}{1-h} d(TSx_0, Tx_0)$$

Since P is a normal cone we obtain

$\lim_{m,n \to \infty} d(TS^m x_0, TS^n x_0) = 0$

Hence, the fact that (X, d) is a complete cone metric space, imply that $(TS^n x_0)$ is a Cauchy sequence in X, therefore is $y_0 \in M$ such that

$\lim_{n \to \infty} TS^n x_0 = y_0$

(iii) If T is sub-sequentially convergent, $\{S^n x_0\}$ has a convergent subsequence, so there is $z_0 \in$ M and $\{n_k\}_{k=1}^{\infty}$ such that

$\lim_{k \to \infty} S^{n_k} x_0 = z_0$

(iv) Since T and S are continuous mappings we obtain:

$\lim_{k \to \infty} T S^{n_k} x_0 = Tz_0$

$\lim_{k \to \infty} T S^{n_k+1} x_0 = TSz_0$

Therefore, $Tz_0 = y_0 = TSz_0$,

Since, T is one to one, then $Sz_0 = z_0$.

So S has a fixed point.

Now, suppose that $Sz_0 = z_0$ and $Sz_1 = z_1$.

$d(TSz_0, TSz_1) \leq \lambda[d(Tz_0, TSz_0) + d(Tz_1, TSz_1)]$

$d(Tz_0, Tz_1) \leq \lambda[d(Tz_0, Tz_0) + d(Tz_1, Tz_1)]$

$d(Tz_0, Tz_1) \leq 0$

$\Longrightarrow d(Tz_0, Tz_1) = 0$

$\Longrightarrow Tz_0 = Tz_1$

Since T is one to one, then we obtain that $z_0 = z_1$.

(v) It is clear that if T is sequentially convergent, then for each $x_0 \in X$, the iterate sequence $\{S^n x_0\}$ converges to z_0.

Theorem 3.3: Let E be a real Banach space, K be a closed, convex subset of E and $T, S: K \to K$

be two mappings such that T is continuous, one-to-one, sub-sequentially convergent with

$$\|TSx - TSy\| \le b[\|Tx - TSy\| + \|Ty - TSx\| - \|TSy - TSx\|] \quad \dots (3.4)$$

Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined as in (2.1) where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.

Proof: By Lemma (3.1), S is a T−Zamfirescu mapping, and by the theorem (3.2) we get that S has a unique fixed point, say x^* in K.

Since S is a T−Zamfirescu mapping, therefore, there is a real number $0 \le k < 1$ such that

$$\|TSx - TSy\| \le k\|Tx - Ty\| \quad \dots (3.5)$$

Let $\{Tx_n\}_{n=0}^{\infty} \in K$ be the T- Mann iteration associated to S defined by (2.1) and $x_0 \in K$ .Then

$$\|Tx_{n+1} - Tx^*\| = \|(1-\alpha_n)Tx_n + \alpha_n TSx_n - Tx^*\|$$
$$= \|(1-\alpha_n)(Tx_n - Tx^*) + \alpha_n(TSx_n - Tx^*)\|$$

Which gives

$$\|Tx_{n+1} - Tx^*\| \le (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n\|TSx_n - Tx^*\| \quad \dots (3.6)$$

Taking $x = x^*$ and $y = x_n$ in (3.5) we get

$$\|TSx^* - TSx_n\| \le k\|Tx^* - Tx_n\|$$

Which implies

$$\|Tx^* - TSx_n\| \le k\|Tx^* - Tx_n\| \quad \dots (3.7)$$

Using (3.6) and (3.7) we obtain,

$$\|Tx_{n+1} - Tx^*\| \le (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n k\|Tx^* - Tx_n\|$$
$$= (1 - \alpha_n + \alpha_n k)\|Tx_n - Tx^*\|$$
$$= [1 - \alpha_n(1-k)]\|Tx_n - Tx^*\|$$

Since $0 \le k < 1, \alpha_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, by setting $\alpha_n = (1-k)\alpha_n$, $r_n = \|Tx_n - Tx^*\|$ and by applying Lemma (2.11) we get that

$$\lim_{n\to\infty} \|Tx_n - Tx^*\| = 0$$

Hence $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where $x^*$is the fixed point of S.

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